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Optimization over multi-order cones

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Optimization over multi-order cones

Baha M. Alzalg* and K. A. Ariyawansa†

Abstract

In this paper we propose multi-order cone programs (MOCPs) as a new class of convex nonlinear optimization problems that includes linear programs, (convex) quadratic programs, second-order cone programs and, more generally, p^{th}-order cone programs as special cases. In MOCPs we minimize a linear objective function over the intersection of an affine set and a product of multi-order cones. We refer to them as deterministic multi-order cone programs (DMCOPs) since data defining them are deterministic. We present the definition of DMOCPs in primal and dual standard forms. Then we introduce two-stage stochastic multi-order cone programs (SMOCPs) (with recourse) to handle uncertainty in data defining DMOCPs and deterministic mixed integer multi-order cone programs (DMIMOCPs) to handle DMOCPs with integer-valued variables. We describe an applicational setting and present DMOCP, SMOCP, and DMIMOCP models arising in that setting.

Keywords: Linear programming; Stochastic programming; Recourse; Second-order cone programming; Mixed integer programming

1 Introduction

Semidefinite programming [16, 13] problems were extensively studied during the late 1990s as a class of optimization problems. They are extensions of linear programs and provide novel modeling capabilities. Interior point algorithms could be derived for them (often utilizing their symbolic similarities to linear programs).

Ariyawansa and Zhu [5] (see also [10]) presented stochastic semidefinite programs that extended stochastic linear programs [17, 6], and allowed the derivation of elegant interior point algorithms [4, 10].

It soon became apparent [9, 1] that almost all applications of semidefinite programs indeed lead to a subset of semidefinite programs termed *second order cone programs.* We refer to them as *deterministic* second order cone programs (DSOCPs) because they are defined using deterministic data. In DSOCP we minimize a linear function over the intersection of an affine set and a Cartesian product of second order cones.

In this paper, we present three extensions of DSOCPs. First, we present primal and dual forms of *(deterministic) multi-order cone programs (DMOCPs)* in which we minimize a linear function over a Cartesian product of p^{th}-order cones (we allow different p values for different cones in the

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product). We present generic applications that extend those in [1, Section 2.2]. We also present a glimpse of the duality theory that we are developing in [2] for DMOCPs.

Second, we present two-stage *stochastic multi-order cone programs* (SMOCPs). SMOCPs are a way of handling uncertainty in data defining DMOCPs. Then we demonstrate that stochastic linear programs and stochastic quadratic programs are special cases of SMOCPs.

Our third extension is introduced to handle modeling situation in which some of the variables in an optimization problem can only take integer values, or even 0 or 1. This leads to (deterministic) *mixed integer multi order cone Programs* (DMIMOCPs) and 0-1 deterministic multi order cone programs (0-1DMOCPs).

We then demonstrate how decision making problems associated with facility location problems lead to a DMOCP model, an SMOCP model, a 0-1DMOCP model, and a DMIMOP model.

We begin with an introduction to our notation.

1.1 Notations

We begin by introducing some notations we use in the sequel.

Let $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{n\vee n}$ denote the vector spaces of real $m\times n$ matrices and real symmetric $n\times n$ matrices respectively. For $U, V \in \mathbb{R}^{n\vee n}$, we write $U \succeq 0$ ($U \succ 0$) to mean that U is positive semidefinite (positive definite), and $U \succeq V$ or $V \preceq U$ to mean that $U - V \succeq 0$.

All vectors we use are column vectors with superscript T indicating transposition. We use "," for adjoining vectors and matrices in a row, and use ";" for adjoining them in a column. So, for example, if $\boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$ are vectors, we have:

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = (\boldsymbol{x}^{\mathsf{T}}, \boldsymbol{y}^{\mathsf{T}}, \boldsymbol{z}^{\mathsf{T}})^{\mathsf{T}} = (\boldsymbol{x}; \boldsymbol{y}; \boldsymbol{z}).$$

If $\mathcal{A} \subseteq \mathbb{R}^k$ and $\mathcal{B} \subseteq \mathbb{R}^l$, then the Cartesian product of $\mathcal{A}\times\mathcal{B} := \{(\boldsymbol{x}; \boldsymbol{y}) : \boldsymbol{x} \in \mathcal{A} \text{ and } \boldsymbol{y} \in \mathcal{B}\}$.

For each vector $\boldsymbol{x} \in \mathbb{R}^k$ indexed from 0, we write $\bar{\boldsymbol{x}}$ for the sub-vector consisting of entries 1 through $k-1$; therefore $\boldsymbol{x} = (x_0; \bar{\boldsymbol{x}})$.

Given $p \geq 1$, the p^{th}-order cone of dimension n is defined as $\mathcal{Q}_p^n := \{\boldsymbol{x} = (x_0; \bar{\boldsymbol{x}}) \in \mathbb{R}\times\mathbb{R}^{n-1} : x_0 \geq ||\bar{\boldsymbol{x}}||_p\}$ where $||\cdot||_p$ denotes the p-norm. The cone $\mathcal{Q}_p^n$ is convex, pointed, closed and with a nonempty interior (see, for example, [18]). As special cases, when $p = 2$ we obtain the second-order cone (also known as the quadratic, Lorentz, or the ice-cream cone) of dimension n, and when $p = 1$ or ∞, $\mathcal{Q}_p^n$ is a polyhedral cone.

We write $\boldsymbol{x} \succeq_{\langle p\rangle}^{\langle n\rangle} \boldsymbol{0}$ to mean that $\boldsymbol{x} \in \mathcal{Q}_p^n$, and $\boldsymbol{x} \succeq_{\langle p\rangle}^{\langle n\rangle} \boldsymbol{y}$ to mean that $\boldsymbol{x} - \boldsymbol{y} \succeq_{\langle p\rangle}^{\langle n\rangle} \boldsymbol{0}$.

Given $1 \leq p_i \leq \infty$ for $i = 1, 2, \cdots, r$. Let $\mathcal{Q}_{\langle p_1,p_2,\cdots,p_r\rangle}^{\langle n_1,n_2,\cdots,n_r\rangle} := \mathcal{Q}_{p_1}^{n_1}\times\mathcal{Q}_{p_2}^{n_2}\times\cdots\times\mathcal{Q}_{p_r}^{n_r}$. We write $\boldsymbol{x} \succeq_{\langle p_1,p_2,\cdots,p_r\rangle}^{\langle n_1,n_2,\cdots,n_r\rangle} \boldsymbol{0}$ to mean that $\boldsymbol{x} \in \mathcal{Q}_{\langle p_1,p_2,\cdots,p_r\rangle}^{\langle n_1,n_2,\cdots,n_r\rangle}$ and $\boldsymbol{x} \succeq_{\langle p_1,p_2,\cdots,p_r\rangle}^{\langle n_1,n_2,\cdots,n_r\rangle} \boldsymbol{y}$ to mean that $\boldsymbol{x} - \boldsymbol{y} \succeq_{\langle p_1,p_2,\cdots,p_r\rangle}^{\langle n_1,n_2,\cdots,n_r\rangle} \boldsymbol{0}$.

It is immediately seen that, for every vector $\boldsymbol{x} \in \mathbb{R}^n$ where $n = \sum_{i=1}^r n_i$, $\boldsymbol{x} \succeq_{\langle p_1,p_2,\cdots,p_r\rangle}^{\langle n_1,n_2,\cdots,n_r\rangle} \boldsymbol{0}$ if and only if $\boldsymbol{x}$ is partitioned conformally as $\boldsymbol{x} = (\boldsymbol{x_1}; \boldsymbol{x_2}; \cdots; \boldsymbol{x_r})$ and $\boldsymbol{x_i} \succeq_{\langle p_i\rangle}^{\langle n_i\rangle} \boldsymbol{0}$ for $i = 1, 2, \cdots, r$. For simplicity, we write:

- $\mathcal{Q}_p^n$ as $\mathcal{Q}_p$ and $\boldsymbol{x} \succeq_{\langle p\rangle}^{\langle n\rangle} 0$ as $\boldsymbol{x} \succeq_{\langle p\rangle} \boldsymbol{0}$ when n is known from the context;

- $\mathcal{Q}^{\langle n_1,n_2,\cdots,n_r\rangle}_{\langle p_1,p_2,\cdots,p_r\rangle}$ as $\mathcal{Q}_{\langle p_1,p_2,\cdots,p_r\rangle}$ and $\boldsymbol{x} \succeq^{\langle n_1,n_2,\cdots,n_r\rangle}_{\langle p_1,p_2,\cdots,p_r\rangle} \mathbf{0}$ as $\boldsymbol{x} \succeq_{\langle p_1,p_2,\cdots,p_r\rangle} \mathbf{0}$ when $n_1, n_2, \cdots, n_r$ are known from the context;

- $\boldsymbol{x} \succeq_{\langle \underbrace{p, p, \cdots, p}_{r \text{ times}} \rangle} \mathbf{0}$ as $\boldsymbol{x} \succeq_{r\langle p\rangle} \mathbf{0}$;

- $\boldsymbol{x} \succeq_{\langle 2\rangle} \mathbf{0}$ as $\boldsymbol{x} \succeq \mathbf{0}$; and $\boldsymbol{x} \succeq_{r\langle 2\rangle} \mathbf{0}$ as $\boldsymbol{x} \succeq_{r} \mathbf{0}$ when the problem includes only (linear and) second-order cone constraints.

Note that, for every vector $\boldsymbol{x} \in \mathbb{R}^n = \overbrace{\mathbb{R}^k \times \mathbb{R}^k \times \cdots \times \mathbb{R}^k}^{r \text{ times}}$, $\boldsymbol{x} \succeq_{r\langle p\rangle} \mathbf{0}$ implies that $\boldsymbol{x}$ is partitioned regularly as $\boldsymbol{x} = (\boldsymbol{x_1}; \boldsymbol{x_2}; \cdots; \boldsymbol{x_r})$ and each subvector $\boldsymbol{x}_i$ lies in the p^{th}-order cone of dimension $k = n/r$ for $i = 1, 2, \cdots, r$. In this case, if $n = r$ and $p = 2$, then $x_i \in \mathcal{Q}^1_2 = \{t \in \mathbb{R} : t \geq 0\}$ for each $i = 1, 2, \cdots, n$. So $\boldsymbol{x} \succeq_n \mathbf{0}$ means the same as $\boldsymbol{x} \geq \mathbf{0}$, i.e., $\boldsymbol{x}$ lies in the nonnegative orthant of $\mathbb{R}^n$.

2 Definition of a DMOCP

Let $r \geq 1$ be an integer, and $p_1, p_2, \cdots, p_r$ are such that $1 \leq p_i \leq \infty$ for $i = 1, 2, \cdots, r$. Let $m, n, n_1, n_2, \cdots, n_r$ be positive integers such that $n = \sum_{i=1}^r n_i$. Then we define a DMOCP in *primal standard form* as

$$(P) \qquad \begin{array}{lrcl} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\ \text{s.t.} & A\,\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \succeq_{\langle p_1,p_2,\cdots,p_r\rangle} & \mathbf{0} \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ constitute given data, $\boldsymbol{x} \in \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ is the decision variable. We define a DMOCP in *dual standard form* as

$$(D) \qquad \begin{array}{lrcl} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{z} & = & \boldsymbol{c} \\ & \boldsymbol{z} & \succeq_{\langle q_1,q_2,\cdots,q_r\rangle} & \mathbf{0} \end{array}$$

where $\boldsymbol{y} \in \mathbb{R}^m$ and $\boldsymbol{z} \in \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ are the decision variables, $q_1, q_2, \cdots, q_r$ are integers such that $1 \leq q_i \leq \infty$ for $i = 1, 2, \cdots, r$.

If (P) and (D) are defined by the same data, and q_i is conjugate to p_i, in the sense that $1/p_i + 1/q_i = 1$ for $i = 1, 2, \cdots, r$, then we can prove relations between (P) and (D) (see Subsection 2.2) justify referring to (D) as the dual of (P) and vice versa.

2.1 Special cases of DMOCPs

A *deterministic p^{th}-order cone programming* (see also [12]) (DPOCP) problem in *primal standard form* is

$$\begin{array}{lrcl} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\ \text{s.t.} & A\,\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \succeq_{r\langle p\rangle} & \mathbf{0} \end{array} \tag{1}$$

where $m, n, n_1, n_2, \cdots, n_r$ are positive integers such that $n = \sum_{i=1}^{r} n_i$, $p \in [1, \infty]$, $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ constitute given data, $\boldsymbol{x} \in \mathbb{R}^n$ is the primal variable. Clearly, DSOCPs are a special case of DMOCPs with $p_i = p \geq 1$ for all $i = 1, 2, \cdots, r$. According to (D), the *dual* problem associated with DPOCP (1) is

$$\begin{array}{lrcl} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{z} & = & \boldsymbol{c} \\ & \boldsymbol{z} & \succeq_{r\langle q\rangle} & \boldsymbol{0} \end{array} \tag{2}$$

where $\boldsymbol{y} \in \mathbb{R}^m$ and $\boldsymbol{z} \in \mathbb{R}^n$ are the dual variables and q is conjugate to p.

Deterministic second-order cone programs (DSOCPs) are a special case of DPOCPs (and hence of DMOCPs) which occurs when $p = 2$ in (1) (and hence $q = 2$ in (2)). A DSOCP problem in *primal standard form* (see [1]) is

$$\begin{array}{lrcl} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} & & \\ \text{s.t.} & A\,\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \succeq_{r} & \boldsymbol{0} \end{array} \tag{3}$$

and its *dual* problem (see [1])

$$\begin{array}{lrcl} \max & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} & & \\ \text{s.t.} & A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{z} & = & \boldsymbol{c} \\ & \boldsymbol{z} & \succeq_{r} & \boldsymbol{0} \end{array} \tag{4}$$

where $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ constitute given data, $\boldsymbol{x} \in \mathbb{R}^n$ is the primal variable, and $\boldsymbol{y} \in \mathbb{R}^m$ and $\boldsymbol{z} \in \mathbb{R}^n$ are the dual variables.

Since DSOCP is a special case of DMOCP, all problems that can be formulated as DSOCPs, such as DLPs (the DSOCP problems (3) and (4) reduce to DLP problems when $r = n$), strictly convex deterministic quadratic programs (DQPs), convex quadratically constrained quadratic programs (QCQPs), and problems with hyperbolic constraints (see [1, 9]) are special cases of DSOCPs and of DMOCPs. The survey paper of Lobo, *et al.* [9] discusses DSOCPs with a number of applications in many areas including a variety of engineering applications.

2.1.1 Examples: Norm minimization problems

In [1] Alizadeh and Goldfarb presented DSOCP formulations of three norm minimization problems where the norm is the Euclidean norm. In this subsection we show how extensions of these three problems where we use arbitrary p norms lead to DMOCPs. Let $\boldsymbol{v}_i = A_i\boldsymbol{x} + \boldsymbol{b}_i \in \mathbb{R}^{n_i-1}$, $i = 1, 2, \cdots, r$. The following norm minimization problems can be cast as DMOCPs:

1. Minimization of the sum of norms:
   The problem $\min \sum_{i=1}^{r} ||\boldsymbol{v}_i||_{p_i}$ can be formulated as

$$\begin{array}{ll} \min & \sum_{i=1}^{r} t_i \\ \text{s.t.} & A_i\boldsymbol{x} + \boldsymbol{b}_i = \boldsymbol{v}_i,\ i = 1, 2, \cdots, r \\ & (t_1; \boldsymbol{v}_1; t_2; \boldsymbol{v}_2; \cdots; t_r; \boldsymbol{v}_r) \ \succeq_{\langle p_1, p_2, \cdots, p_r\rangle} \boldsymbol{0} \end{array}$$

2. Minimization of the maximum of norms:
   The problem min $\max_{1\le i\le r}\ ||\boldsymbol{v}_i||_{p_i}$ can be expressed as the DMOCP problem

$$\begin{array}{ll} \min & t \\ \text{s.t.} & A_i\boldsymbol{x}+\boldsymbol{b}_i=\boldsymbol{v}_i,\ i=1,2,\cdots,r \\ & (t;\boldsymbol{v}_1;t;\boldsymbol{v}_2;\cdots;t;\boldsymbol{v}_r)\ \succeq_{\langle p_1,p_2,\cdots,p_r\rangle}\mathbf{0} \end{array}$$

3. Minimization of the sum of the k largest norms:
   More generally, the problem of minimizing the sum of the k largest norms can also be cast as DMOCPs. Let the norms $||\boldsymbol{v}_{[1]}||_{p_{[1]}}, ||\boldsymbol{v}_{[2]}||_{p_{[2]}},\cdots,||\boldsymbol{v}_{[r]}||_{p_{[r]}}$ be the norms $||\boldsymbol{v}_1||_{p_1}, ||\boldsymbol{v}_2||_{p_2},\cdots,||\boldsymbol{v}_r||_{p_r}$ sorted in nonincreasing order. Then the problem $\sum_{i=1}^{r}||\boldsymbol{v}_{[i]}||_{p_{[i]}}$ can be formulated as (see also [1] or [9] and the related references contained therein)

$$\begin{array}{ll} \min & \sum_{i=1}^{r}s_i+kt \\ \text{s.t.} & A_i\boldsymbol{x}+\boldsymbol{b}_i=\boldsymbol{v}_i,\ i=1,2,\cdots,r \\ & (s_1+t;\boldsymbol{v}_1;s_2+t;\boldsymbol{v}_2;\cdots;s_r+t;\boldsymbol{v}_r)\ \succeq_{\langle p_1,p_2,\cdots,p_r\rangle}\mathbf{0} \\ & s_i\ge 0 \end{array}$$

2.2 Duality

Since DMOCPs are a class of convex optimization problems, we can develop a duality theory for them. A forthcoming paper [2] presents such a duality theory. Here we indicate weak and strong duality for the pair (P, D) as justification for referring to them as a primal dual pair.

We first show that the dual of the p^{th}-order cone of dimension n is the q^{th}-order cone of dimension n, where q is the conjugate to p. For any cone $\mathcal{K}$, the dual cone $\mathcal{K}^*$ is defined by $\mathcal{K}^*:=\{\boldsymbol{y}\in\mathbb{R}^n:\boldsymbol{y}^{\mathsf{T}}\boldsymbol{x}\ge 0,\forall\boldsymbol{x}\in\mathcal{K}\}$.

Lemma 1 ${\mathcal{Q}_p}^*=\mathcal{Q}_q$, where $1\le p\le\infty$ and q is the conjugate to p. More generally, ${\mathcal{Q}_{\langle p_1,p_2,\cdots,p_r\rangle}}^*=\mathcal{Q}_{\langle q_1,q_2,\cdots,q_r\rangle}$, where $1\le p_i\le\infty$ and q_i is the conjugate to p_i for $i=1,2,\cdots,r$.
Proof. We assume that $\mathcal{Q}_p\subset\mathbb{R}^n$. The proof of the second part trivially follows from the first part. To prove the first part, we first prove that $\mathcal{Q}_q\subseteq{\mathcal{Q}_p}^*$. Let $\boldsymbol{x}=(x_0;\bar{\boldsymbol{x}})\in\mathcal{Q}_q$, we show that $\boldsymbol{x}\in{\mathcal{Q}_p}^*$ by verifying that $\boldsymbol{x}^{\mathsf{T}}\bar{\boldsymbol{y}}\ge 0$ for any $\boldsymbol{y}\in\mathcal{Q}_p$. So let $\boldsymbol{y}=(y_0;\bar{\boldsymbol{y}})\in\mathcal{Q}_p$. Then $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{y}=x_0\ y_0+\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\ge||\bar{\boldsymbol{x}}||_q||\bar{\boldsymbol{y}}||_p+\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\ge|\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}|+\bar{\boldsymbol{x}}^{\mathsf{T}}\bar{\boldsymbol{y}}\ge 0$, where the first inequality follows from the fact that $\boldsymbol{x}\in\mathcal{Q}_q$ and $\boldsymbol{y}\in\mathcal{Q}_p$ and the second one from Hölder's inequality. Now we show ${\mathcal{Q}_p}^*\subseteq\mathcal{Q}_q$. Let $\boldsymbol{y}=(y_0;\bar{\boldsymbol{y}})\in{\mathcal{Q}_p}^*$, we show that $\boldsymbol{y}\in\mathcal{Q}_q$ by verifying that $y_0\ge||\bar{\boldsymbol{y}}||_q$. This is trivial if $\bar{\boldsymbol{y}}=\mathbf{0}$ or $p=\infty$. If $\bar{\boldsymbol{y}}\ne\mathbf{0}$ and $1\le p<\infty$, let $\boldsymbol{u}:=({y_1}^{p/q};{y_2}^{p/q};\cdots;{y_{n-1}}^{p/q})$ and consider $\boldsymbol{x}:=(||\boldsymbol{u}||_p;-\boldsymbol{u})\in\mathcal{Q}_p$. Then by using Hölder's inequality, where the equality is attained, we obtain $0\le\boldsymbol{x}^{\mathsf{T}}\boldsymbol{y}=||\boldsymbol{u}||_p\ y_0-\boldsymbol{u}^{\mathsf{T}}\bar{\boldsymbol{y}}=||\boldsymbol{u}||_p\ y_0-||\boldsymbol{u}||_p||\bar{\boldsymbol{y}}||_q=||\boldsymbol{u}||_p\ (y_0-||\bar{\boldsymbol{y}}||_q)$. This gives that $y_0\ge||\bar{\boldsymbol{y}}||_q$. $\square$

It follows from this lemma that the second-order cone is self-dual, i.e., ${\mathcal{Q}_2}^*=\mathcal{Q}_2$. From this lemma we also deduce that the p^{th}-order cone is reflexive, i.e., ${\mathcal{Q}_p}^{**}=\mathcal{Q}_p$, and more generally, also $\mathcal{Q}_{\langle p_1,p_2,\cdots,p_r\rangle}$ is reflexive. On the basis of this fact, it is natural to infer that the dual of the dual is the primal. Using the above lemma, we can prove the following weak duality property.

Theorem 1 *(Weak duality)* If $\boldsymbol{x}$ is any primal feasible solution of (P) and $(\boldsymbol{y}, \boldsymbol{z})$ is any dual feasible solution of (D), then the duality gap $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathsf{T}}\boldsymbol{z} \geq 0$.

Proof. Note that $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} = (A^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{z})^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} = \boldsymbol{y}^{\mathsf{T}}A\boldsymbol{x} + \boldsymbol{z}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{y}^{\mathsf{T}}\boldsymbol{b} = \boldsymbol{y}^{\mathsf{T}}(A\boldsymbol{x} - \boldsymbol{b}) + \boldsymbol{z}^{\mathsf{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathsf{T}}\boldsymbol{z}$. Since $\boldsymbol{x} \in \mathcal{Q}_{\langle p_1, p_2, \cdots, p_r\rangle}$ and $\boldsymbol{z} \in \mathcal{Q}_{\langle q_1, q_2, \cdots, q_r\rangle} = \mathcal{Q}_{\langle p_1, p_2, \cdots, p_r\rangle}{}^{*}$, we conclude that $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{z} \geq 0$. $\square$

We can now apply the duality relations [11, Theorem 4.2.1] to obtain:

Theorem 2 *(Strong duality)* Assume the interior of (P) to be not empty and the objective value of (P) to be bounded below on the feasible region. Then (D) is solvable and the optimal objective value p^* of (P) and the dual objective value d^* satisfy the relation $p^* = d^*$.

3 Definition of an SMOCP

In this section we define two-stage *stochastic multi-order cone programs* (SMCOPs) with recourse based on DMOCP (P) analogous to the way SLPs is defined based on DLPs. Let $r_1, r_2 \geq 1$ be integers. For $i = 1, 2, \cdots, r_1$ and $j = 1, 2, \cdots, r_2$, let $p_{1i}, p_{2j} \in [1, \infty]$ and $m_1, m_2, n_1, n_2, n_{1i}, n_{2j}$ be positive integers such that $n_1 = \sum_{i=1}^{r_1} n_{1i}$ and $n_2 = \sum_{i=1}^{r_2} n_{2j}$. An SMOCP with recourse in primal standard form is defined based on deterministic data $A \in \mathbb{R}^{m_1 \times n_1}, \boldsymbol{b} \in \mathbb{R}^{m_1}$ and $\boldsymbol{c} \in \mathbb{R}^{n_1}$ and random data $T \in \mathbb{R}^{m_2 \times n_1}, W \in \mathbb{R}^{m_2 \times n_2}, \boldsymbol{h} \in \mathbb{R}^{m_2}$ and $\boldsymbol{d} \in \mathbb{R}^{n_2}$ whose realizations depend on an underlying outcome ω in an event space Ω with a known probability function P. Given this data, an SMOCP with recourse in *primal standard form* is

$$
\begin{array}{ll}
\min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}\,[Q(\boldsymbol{x}, \omega)] \\
\text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\
 & \boldsymbol{x} \succeq_{\langle p_{11}, p_{12}, \cdots, p_{1r_1}\rangle} \boldsymbol{0}
\end{array}
\tag{5}
$$

where $\boldsymbol{x} \in \mathbb{R}^{n_1} = \mathbb{R}^{n_{11}} \times \mathbb{R}^{n_{12}} \times \cdots \times \mathbb{R}^{n_{1r_1}}$ is the first-stage decision variable and $Q(\boldsymbol{x}, \omega)$ is the minimum of the problem

$$
\begin{array}{ll}
\min & \boldsymbol{d}(\omega)^{\mathsf{T}}\boldsymbol{y} \\
\text{s.t.} & T(w)\boldsymbol{x} + W(\omega)\boldsymbol{y} = \boldsymbol{h}(\omega) \\
 & \boldsymbol{y} \succeq_{\langle p_{21}, p_{22}, \cdots, p_{2r_2}\rangle} \boldsymbol{0}
\end{array}
\tag{6}
$$

where $\boldsymbol{y} \in \mathbb{R}^{n_2} = \mathbb{R}^{n_{21}} \times \mathbb{R}^{n_{22}} \times \cdots \times \mathbb{R}^{n_{2r_2}}$ is the second-stage variable and

$$
\mathbb{E}[Q(\boldsymbol{x}, \omega)] := \int_{\Omega} Q(\boldsymbol{x}, \omega) P(d\omega).
$$

3.1 Special cases of SMOCPs

In this part we present some important special cases of SMOCPs. *Stochastic p^{th}-order cone programs* (SPOCPs) are a special case of SMOCPs which occurs when $p_{1i} = p_{2j} = p \geq 1$ for all $i = 1, 2, \cdots, r_1$ and $j = 1, 2, \cdots, r_2$ in (5, 6). An SPOCP problem therefore is

$$
\begin{array}{ll}
\min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}\,[Q(\boldsymbol{x}, \omega)] \\
\text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\
 & \boldsymbol{x} \succeq_{r_1 \langle p\rangle} \boldsymbol{0}
\end{array}
\tag{7}
$$

where $\boldsymbol{x} \in \mathbb{R}^{n_1} = \mathbb{R}^{n_{11}} \times \mathbb{R}^{n_{12}} \times \cdots \times \mathbb{R}^{n_{1r_1}}$ is the first-stage decision variable and $Q(\boldsymbol{x}, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \boldsymbol{d}(\omega)^{\mathsf{T}}\boldsymbol{y} \\ \text{s.t.} & T(w)\boldsymbol{x} + W(\omega)\boldsymbol{y} = \boldsymbol{h}(\omega) \\ & \boldsymbol{y} \succeq_{r_2\langle p\rangle} \boldsymbol{0} \end{array} \tag{8}$$

where $\boldsymbol{y} \in \mathbb{R}^{n_2} = \mathbb{R}^{n_{21}} \times \mathbb{R}^{n_{22}} \times \cdots \times \mathbb{R}^{n_{2r_2}}$ is the second-stage variable and

$$\mathbb{E}[Q(\boldsymbol{x}, \omega)] := \int_{\Omega} Q(\boldsymbol{x}, \omega) P(d\omega).$$

Stochastic second-order cone programs (SSOCPs) are a special case of SPOCPs (and hence of SMOCPs) which occurs when $p = 2$ in (7, 8). An SSOCP problem (see also [10]) is

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}\left[Q(\boldsymbol{x}, \omega)\right] \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq_{r_1} \boldsymbol{0} \end{array} \tag{9}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_1} = \mathbb{R}^{n_{11}} \times \mathbb{R}^{n_{12}} \times \cdots \times \mathbb{R}^{n_{1r_1}}$ is the first-stage decision variable and $Q(\boldsymbol{x}, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & \boldsymbol{d}(\omega)^{\mathsf{T}}\boldsymbol{y} \\ \text{s.t.} & T(w)\boldsymbol{x} + W(\omega)\boldsymbol{y} = \boldsymbol{h}(\omega) \\ & \boldsymbol{y} \succeq_{r_2} \boldsymbol{0} \end{array} \tag{10}$$

where $\boldsymbol{y} \in \mathbb{R}^{n_2} = \mathbb{R}^{n_{21}} \times \mathbb{R}^{n_{22}} \times \cdots \times \mathbb{R}^{n_{2r_2}}$ is the second-stage variable and

$$\mathbb{E}[Q(\boldsymbol{x}, \omega)] := \int_{\Omega} Q(\boldsymbol{x}, \omega) P(d\omega).$$

Note that when $r_1 = n_1$ and $r_2 = n_2$, SSOCP (9, 10) reduces to an SLP. So SLPs are a special case of SSOCPs.

In the rest of this section, we will show that stochastic quadratic programs (SQPs) can also be cast as SSOCPs. Our proof is parallel to the proof of the fact that DQPs is a subclass of DSOCPs (see [1]). Recall that a two-stage SQP (with recourse) is defined based on deterministic data $C \in \mathbb{R}^{n_1 \vee n_1}, C \succ 0, \boldsymbol{c} \in \mathbb{R}^{n_1}, A \in \mathbb{R}^{m_1 \times n_1}$ and $\boldsymbol{b} \in \mathbb{R}^{m_1}$; and random data $H \in \mathbb{R}^{n_2 \vee n_2}, H \succ 0, \boldsymbol{d} \in \mathbb{R}^{n_2}, T \in \mathbb{R}^{m_2 \times n_1}, W \in \mathbb{R}^{m_2 \times n_2}$, and $\boldsymbol{h} \in \mathbb{R}^{m_2}$ whose realizations depend on an underlying outcome in an event space Ω with a known probability function P. Given this data, an SQP with recourse is

$$\begin{array}{ll} \min & q_1(\boldsymbol{x}, \omega) = \boldsymbol{x}^{\mathsf{T}}C\boldsymbol{x} + \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}[Q(\boldsymbol{x}, \omega)] \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{11}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_1}$ is the first-stage decision variable, $Q(\boldsymbol{x}, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & q_2(\boldsymbol{y}, \omega) = \boldsymbol{y}^{\mathsf{T}}H(\omega)\boldsymbol{y} + \boldsymbol{d}(\omega)^{\mathsf{T}}\boldsymbol{y} \\ \text{s.t.} & T(\omega)\boldsymbol{x} + W(\omega)\boldsymbol{y} = \boldsymbol{h}(\omega) \\ & \boldsymbol{y} \geq \boldsymbol{0} \end{array} \tag{12}$$

where $\boldsymbol{y} \in \mathbb{R}^{n_2}$ is the second-stage variable, and

$$\mathbb{E}[Q(\boldsymbol{x}, \omega)] := \int_{\Omega} Q(\boldsymbol{x}, \omega) P(d\omega).$$

Observe that the objective function of (11) can be written as (see §2 in [1]),

$$q_1(\boldsymbol{x}_1, \omega) = ||\bar{\boldsymbol{u}}||^2 + \mathbb{E}[Q(\boldsymbol{x}, \omega)] - \frac{1}{4}\boldsymbol{c}^{\mathsf{T}} C^{-1} \boldsymbol{c} \text{ where } \bar{\boldsymbol{u}} = C^{1/2}\boldsymbol{x} + \frac{1}{2} C^{-1/2}\boldsymbol{c}.$$

Similarly, the objective function of (12) can be written as

$$q_2(\boldsymbol{y}, \omega) = ||\bar{\boldsymbol{v}}||^2 - \frac{1}{4}\ \boldsymbol{d}(\omega)^{\mathsf{T}} H(\omega)^{-1} \boldsymbol{d}\ (\omega) \text{ where } \bar{\boldsymbol{v}} = H(\omega)^{1/2}\boldsymbol{y} + \frac{1}{2} H(\omega)^{-1/2}\boldsymbol{d}(\omega).$$

Thus, problem (11, 12) can be transformed into the SSOCP:

$$\begin{array}{ll} \min & u_0 \\ \text{s.t.} & \bar{\boldsymbol{u}} - C^{1/2}\boldsymbol{x} = \frac{1}{2}\ C^{-1/2}\boldsymbol{c} \\ & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0,\ (u_0; \bar{\boldsymbol{u}}) \succeq \boldsymbol{0} \end{array} \tag{13}$$

where $Q(\boldsymbol{x}, \omega)$ is the minimum of the problem

$$\begin{array}{ll} \min & v_0 \\ \text{s.t.} & \bar{\boldsymbol{v}} - H(\omega)^{1/2}\boldsymbol{y} = \frac{1}{2}\ H(\omega)^{-1/2}\boldsymbol{d}(\omega) \\ & T(\omega)\boldsymbol{x} + W(\omega)\boldsymbol{y} = \boldsymbol{h}(\omega) \\ & \boldsymbol{y} \geq \boldsymbol{0};\ (u_0; \bar{\boldsymbol{v}}) \succeq \boldsymbol{0} \end{array} \tag{14}$$

where

$$\mathbb{E}[Q(\boldsymbol{x}, \omega)] := \int_{\Omega} Q(\boldsymbol{x}, \omega) P(d\omega).$$

Note that both problems (the SQP problem and the SSOCP problem) will have the same minimizers, but their optimal objective values are equal up to constants. More precisely, the difference between the optimal objective values of (12) and (14) would be $-\frac{1}{2}\boldsymbol{d}(\omega)^{\mathsf{T}} H(\omega)^{-1}\ \boldsymbol{d}(\omega)$. Consequently, the optimal objective values of (11, 12) and (13, 14) will differ by

$$-\frac{1}{2}\ \boldsymbol{c}^{\mathsf{T}} C^{-1}\boldsymbol{c} - \frac{1}{2}\int_{\Omega} \left(\boldsymbol{d}(\omega)^{\mathsf{T}}\ H(\omega)^{-1}\boldsymbol{d}(\omega)\right) P(d\omega).$$

4 Definitions of a DMIMOCP and a 0-1DMOCP

In this section we introduce two important related problems that result when decision variables in an MOCP can only take integer values. Consider the DMOCP problem (P). If we require an additional constraint that a subset of the variables have to attain 0-1 values, then we are interested in optimization problem of the form

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq_{\langle p_1, p_2, \cdots, p_r \rangle} \boldsymbol{0} \\ & x_k \in \{0, 1\}, \; k \in \Gamma \end{array}$$

where $\Gamma \subseteq \{1, 2, \cdots, n\}$, the decision variable $\boldsymbol{x} \in \mathbb{R}^n$ has some of its components x_k $(k \in \Gamma)$ with integer values and bounded by $\alpha_k, \beta_k \in \mathbb{R}$. This class of optimization problems may be termed as *deterministic 0-1 multi-order cone programs* (0-1DMOCPs).

A more general and interesting problem when in an DMOCP some variables can only take integer values. If we are given the same data $A, \boldsymbol{b}$, and $\boldsymbol{c}$ as in (P), then we are interested in the problem of the form

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \succeq_{\langle p_1, p_2, \cdots, p_r \rangle} \boldsymbol{0} \\ & x_k \in [\alpha_k, \beta_k] \bigcap \mathbb{Z}, \; k \in \Gamma \end{array} \tag{15}$$

where $\Gamma \subseteq \{1, 2, \cdots, n\}$, the decision variable $\boldsymbol{x} \in \mathbb{R}^n$ has some of its components x_k $(k \in \Gamma)$ with integer values and bounded by $\alpha_k, \beta_k \in \mathbb{R}$. This class of optimization problems may be termed as *deterministic mixed integer multi-order cone programs* (DMIMOCPs). The relationships among DMIMOCPs, *deterministic mixed integer p^{th}-order cone programs* (DMIPOCPs) (which occurs when $p_i = p \geq 1$ for all $i = 1, 2, \cdots, r$), deterministic mixed integer second-order cone programs (DMISOCPs) [7] (which occurs when $p_i = 2$ for all $i = 1, 2, \cdots, r$), and deterministic mixed integer linear programs (DMILCPs) (or deterministic mixed integer quadratic programs) are the same as those among DMOCPs, DPOCPs, DSOCPs, and DMILPs (or DMIQPs), respectively.

We can also handle uncertainty in data defining DMIMOCPs by defining two-stage *stochastic mixed integer multi-order cone programs* (SMIMCOPs) with recourse (which generalizes two-stage stochastic mixed integer second-order cone programs [3]) based on DMIMOCP (15) analogous to the way SMOCPs (5, 6) are defined based on DMOCPs (P). See Figure 1 which shows conceptual relationships among the optimization problems over multi-order cones described above and their special cases.

5 An application

Our application is four versions of the *facility location problem* (FLP). For these four versions we present problem descriptions leading to a DMOCP model, an SMOCP model, a 0-1DMOCP model, and a DMIMOCP model.

In FLPs we are interested in choosing a location to build a new facility or locations to build multiple new facilities so that an appropriate measure of distance from the new facilities to existing facilities is minimized. FLPs arise when decisions on locating airports, regional campuses, wireless communications towers, etc. are to be made. There are different ways of classifying FLPs. Following are some of these ways (see also [14]):

- We can classify FLPs based on the number of new facilities in the following sense: if we add only one new facility then we get a problem known as a *single facility location problem*

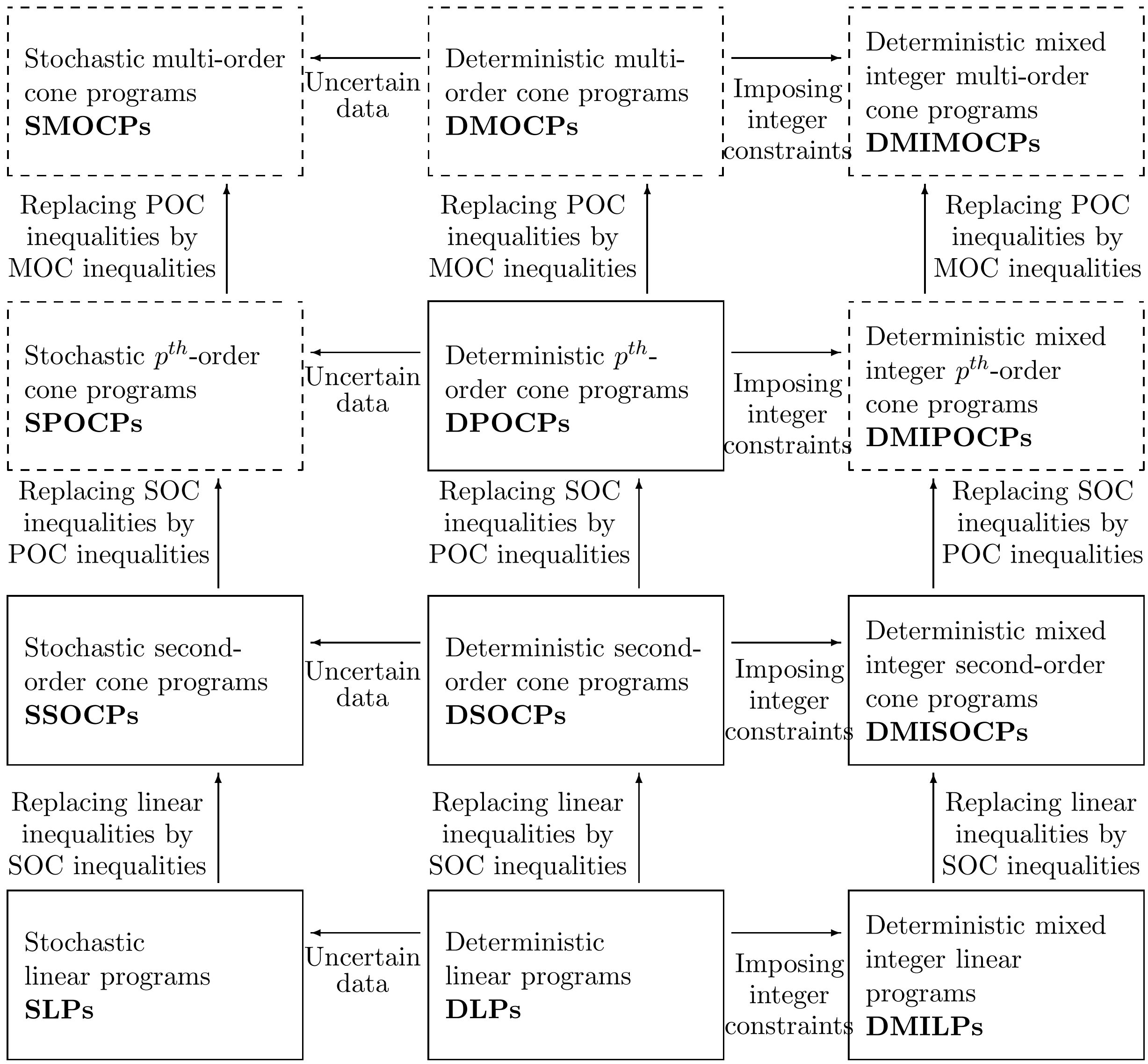

Figure 1: Conceptual relationships among the optimization problems over multi-order cones and their special cases.

(SFLP), while if we add multiple new facilities instead of adding only one, then we get a more general problem known as a *multiple facility location problem* (MFLP).

- Another way of classification is based on the distance measure used in the model between the facilities. If we use the Euclidean distance then these problems are called *Euclidean facility location problems* (EFLPs), if we use the rectilinear distance (also known as L_1 distance, city block distance, or Manhattan distance) then these problems are called *rectilinear facility location problems* (RFLPs). Furthermore, in some applications we use both the Euclidean and the rectilinear distances (based on the relationships between the pairs of facilities) as the distance measures used in the model between the facilities to get a mixed of EFLPs and RFLPs that we refer to as *Euclidean-rectilinear facility location problems* (ERFLPs).

- When the new facilities can be placed any place in solution space, the problem is called a

continuous facility location problem (CFLP), but usually the decision maker needs the new facilities to be placed at *specific locations* (called nodes) and not in any place in the solution space. In this case the problem is called a *discrete facility location problem* (DFLP).

- In some applications, the locations of existing facilities cannot be fully specified because the locations of some of them depend on information not available at the time when decision needs to be made but will only be available at a later point in time. In this case, we are interested in *stochastic facility location problems* (or abbreviated as stochastic FLPs). When the locations of all old facilities are fully specified, FLPs are called *deterministic facility location problems* (or abbreviated as deterministic FLPs).

FLPs have seen a great deal of recent research activity. For further details, consult the book of Tompkins and *et al.* [14]. In particular, deterministic Euclidean facility location problems are often cited as an application of deterministic second-order cone programs (see for example [15] and [8]). Each one of the next subsections is devoted to a version of ERFLPs. Specifically, we consider deterministic continuous Euclidean-rectilinear facility location problems (deterministic CERFLPs) which leads to a DMOCP model, stochastic continuous Euclidean-rectilinear facility location problems (stochastic CERFLPs) which leads to an SMOCP model, deterministic discrete Euclidean-rectilinear facility location problems (deterministic DERFLPs) which leads to a 0-1DMOCP model, and deterministic ERFLPs with integrality constraints which leads to a DMIMOCP model.

5.1 Deterministic CERFLPs—A DMOCP model

In deterministic single ERFLPs, we are interested in choosing a location to build a new facility among existing facilities so that this location minimizes the sum of a weighted (either Euclidean or rectilinear) distance to all existing facilities.

Assume that we are given $r+s$ existing facilities represented by the fixed points $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_r$, $\boldsymbol{a}_{r+1}, \boldsymbol{a}_{r+2}, \cdots, \boldsymbol{a}_{r+s}$ in $\mathbb{R}^n$, and we plan to place a new facility represented by $\boldsymbol{x}$ so that we minimize the weighted sum of the Euclidean distances between $\boldsymbol{x}$ and each of the points $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_r$ and the weighted sum of the rectilinear distances between $\boldsymbol{x}$ and each of the points $\boldsymbol{a}_{r+1}, \boldsymbol{a}_{r+2}, \cdots, \boldsymbol{a}_{r+s}$. This leads us to the problem

$$\min \quad \textstyle\sum_{i=1}^{r} w_i \; ||\boldsymbol{x} - \boldsymbol{a}_i||_2 + \sum_{i=r+1}^{r+s} w_i \; ||\boldsymbol{x} - \boldsymbol{a}_i||_1$$

or, alternatively, to the problem

$$\begin{array}{ll} \min & \sum_{i=1}^{r+s} w_i \; t_i \\ \\ \text{s.t.} & (t_1; \boldsymbol{x} - \boldsymbol{a}_1; \cdots; t_r; \boldsymbol{x} - \boldsymbol{a}_r) \succeq_{r\langle 2\rangle} \mathbf{0} \\ & (t_{r+1}; \boldsymbol{x} - \boldsymbol{a}_{r+1}; \cdots; t_{r+s}; \boldsymbol{x} - \boldsymbol{a}_{r+s}) \succeq_{s\langle 1\rangle} \mathbf{0} \end{array}$$

where w_i is the weight associated with the ith existing facility and the new facility for $i = 1, 2, \ldots, r+s$.

In deterministic multiple ERFLPs we add m new facilities, namely $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m \in \mathbb{R}^n$, instead of adding only one. We have two cases depending whether or not there is an interaction among the new facilities in the underlying model. If there is no interaction between the new facilities, we are just concerned in minimizing the weighted sums of the distance between each one of the new facilities and each one of the fixed facilities. In other words, we solve the following DMOCP model:

$$
\begin{array}{lll}
\min & \sum_{j=1}^{m}\sum_{i=1}^{r+s} w_{ij}\ t_{ij} & \\
\text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{rj}; \boldsymbol{x}_j - \boldsymbol{a}_r) \succeq_{r\langle 2\rangle} \mathbf{0}, \ \ j = 1, 2, \cdots, m & \\
 & (t_{(r+1)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+1}; \cdots; t_{(r+s)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+s}) \succeq_{s\langle 1\rangle} \mathbf{0}, \ \ j = 1, 2, \cdots, m &
\end{array} \tag{16}
$$

where w_{ij} is the weight associated with the ith existing facility and the jth new facility for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, r + s$.

If interaction exists among the new facilities, then, in addition to the above requirements, we need to minimize the sum of the (either Euclidean or rectilinear) distances between each pair of the new facilities. Let $1 \leq l \leq m$ and assume that we are required to minimize the weighted sum of the Euclidean distances between each pair of the new facilities $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_l$ and the weighted sum of the rectilinear distances between each pair of the new facilities $\boldsymbol{x}_{l+1}, \boldsymbol{x}_{l+2}, \cdots, \boldsymbol{x}_m$. In this case, we are interested in a model of the form:

$$
\begin{array}{ll}
\min & \sum_{j=1}^{m}\sum_{i=1}^{r+s} w_{ij}\ t_{ij} + \sum_{j=2}^{m}\sum_{j'=1}^{j-1} \hat{w}_{jj'}\hat{t}_{jj'} \\
\text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{rj}; \boldsymbol{x}_j - \boldsymbol{a}_r) \succeq_{r\langle 2\rangle} \mathbf{0}, \ \ j = 1, 2, \cdots, m \\
 & (t_{(r+1)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+1}; \cdots; t_{(r+s)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+s}) \succeq_{s\langle 1\rangle} \mathbf{0}, \ \ j = 1, 2, \cdots, m \\
 & (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jl}; \boldsymbol{x}_j - \boldsymbol{x}_l) \succeq_{(l-j)\langle 2\rangle} \mathbf{0}, \ \ j = 1, 2, \cdots, l-1 \\
 & (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jm}; \boldsymbol{x}_j - \boldsymbol{x}_m) \succeq_{(m-j)\langle 1\rangle} \mathbf{0}, \ \ j = l+1, 2, \cdots, m-1
\end{array} \tag{17}
$$

where $\hat{w}_{jj'}$ is the weight associated with the new facilities j' and j for $j' = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, m$.

5.2 Stochastic CERFLPs—An SMOCP model

Before we describe the stochastic version of this generic application, we indicate a more concrete version of it. Assume that we have a new city with many suburbs and we want to build a hospital for treating the residents of this city. Some people live in the city at the present time. As the city expands, many houses in new suburbs need to be built and the locations of these suburbs will be known in the future. Our goal is to find the best location of this hospital so that it can serve the current suburbs and the new ones. This location must be determined at the current time and before information about the locations of the new suburbs become available.

Generally speaking, let $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{r_1}, \boldsymbol{a}_{r_1+1}, \boldsymbol{a}_{r_1+2}, \cdots, \boldsymbol{a}_{r_1+s_1}$ be fixed points in $\mathbb{R}^n$ representing the coordinates of r_1+s_1 existing fixed facilities and $\tilde{\boldsymbol{a}}_1(\omega), \tilde{\boldsymbol{a}}_2(\omega), \cdots, \tilde{\boldsymbol{a}}_{r_2}(\omega), \tilde{\boldsymbol{a}}_{r_2+1}(\omega), \tilde{\boldsymbol{a}}_{r_2+2}(\omega), \cdots, \tilde{\boldsymbol{a}}_{r_2+s_2}(\omega)$ be random points in $\mathbb{R}^n$ representing the coordinates of $r_2 + s_2$ random facilities whose realizations depends on an underlying outcome ω in an event space Ω with a known probability function P.

Suppose that at present we do not know the realizations of $r_2 + s_2$ random facilities, and that at some point in time in future the realizations of these $r_2 + s_2$ random facilities become known.

Our goal is to locate m new facilities $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m \in \mathbb{R}^n$ that minimize the the following sums:

- The weighted sums of the Euclidean distance between each one of the new facilities and each one of the fixed facilities $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{r_1}$,
- The weighted sums of the rectilinear distance between each one of the new facilities and each one of the fixed facilities $\boldsymbol{a}_{r_1+1}, \boldsymbol{a}_{r_1+2}, \cdots, \boldsymbol{a}_{r_1+s_1}$,
- The expected weighted sums of the Euclidean distance between each one of the new facilities and each one of the random facilities $\tilde{\boldsymbol{a}}_1(\omega), \tilde{\boldsymbol{a}}_2(\omega), \cdots, \tilde{\boldsymbol{a}}_{r_2}(\omega)$,
- The expected weighted sums of the rectilinear distance between each one of the new facilities and each one of the random facilities $\tilde{\boldsymbol{a}}_{r_2+1}(\omega), \tilde{\boldsymbol{a}}_{r_2+2}(\omega)\ , \cdots, \tilde{\boldsymbol{a}}_{r_2+s_2}(\omega)$.

Note that this decision needs to be made before the realizations of the $r_2 + s_2$ random facilities become available. If there is no interaction between the new facilities, then the DMOCP model (16) becomes the following SMOCP model:

$$
\begin{array}{ll}
\min & \sum_{j=1}^{m} \sum_{i=1}^{r_1+s_1} w_{ij}\ t_{ij} + \mathbb{E}\ [Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega)] \\
\text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{r_1 j}; \boldsymbol{x}_j - \boldsymbol{a}_{r_1}) \succeq_{r_1\langle 2\rangle} \mathbf{0},\ \ j = 1, 2, \cdots, m \\
& (t_{(r_1+1)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r_1+1}; \cdots; t_{(r_1+s_1)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r_1+s_1}) \succeq_{s_1\langle 1\rangle} \mathbf{0},\ \ j = 1, 2, \cdots, m
\end{array}
$$

where $Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega)$ is the minimum of the problem

$$
\begin{array}{ll}
\min & \sum_{j=1}^{m} \sum_{i=1}^{r_2+s_2} \tilde{w}_{ij}(\omega)\ \tilde{t}_{ij} \\
\text{s.t.} & (\tilde{t}_{1j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_1(\omega); \cdots; \tilde{t}_{r_2 j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_{r_2}(\omega)) \succeq_{r_2\langle 2\rangle} \mathbf{0},\ \ j = 1, 2, \cdots, m \\
& (\tilde{t}_{(r_2+1)j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_{(r_2+1)}(\omega); \cdots; \tilde{t}_{(r_2+s_2)j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_{(r_2+s_2)}(\omega)) \succeq_{s_2\langle 1\rangle} \mathbf{0},\ \ j = 1, 2, \cdots, m
\end{array}
$$

and

$$
\mathbb{E}[Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega)] := \int_{\Omega} Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega) P(d\omega).
$$

where w_{ij} is the weight associated with the ith existing facility and the jth new facility for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, r_1 + s_1$, and $\tilde{w}_{ij}(\omega)$ is the weight associated with the ith random existing facility and the jth new facility for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, r_2 + s_2$.

If interaction exists among the new facilities, then the DMOCP model (17) becomes the following SMOCP model:

$$
\begin{array}{ll}
\min & \sum_{j=1}^{m} \sum_{i=1}^{r_1+s_1} w_{ij}\; t_{ij} + \sum_{j=2}^{m} \sum_{j'=1}^{j-1} \hat{w}_{jj'} \hat{t}_{jj'} + \mathbb{E}\left[Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega)\right] \\
\text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{r_1 j}; \boldsymbol{x}_j - \boldsymbol{a}_{r_1}) \succeq_{r_1\langle 2\rangle} \mathbf{0}, \;\; j = 1, 2, \cdots, m \\
& (t_{(r_1+1)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r_1+1}; \cdots; t_{(r_1+s_1)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r_1+s_1}) \succeq_{s_1\langle 1\rangle} \mathbf{0}, \;\; j = 1, 2, \cdots, m \\
& (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jl}; \boldsymbol{x}_j - \boldsymbol{x}_l) \succeq_{(l-j)\langle 2\rangle} \mathbf{0}, \;\; j = 1, 2, \cdots, l-1 \\
& (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jm}; \boldsymbol{x}_j - \boldsymbol{x}_m) \succeq_{(m-j)\langle 1\rangle} \mathbf{0}, \;\; j = l+1, 2, \cdots, m-1
\end{array}
$$

where $Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega)$ is the minimum of the problem

$$
\begin{array}{ll}
\min & \sum_{j=1}^{m} \sum_{i=1}^{r_2+s_2} \tilde{w}_{ij}(\omega)\; \tilde{t}_{ij} + \sum_{j=2}^{m} \sum_{j'=1}^{j-1} \hat{w}_{jj'} \hat{t}_{jj'} \\
\text{s.t.} & (\tilde{t}_{1j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_1(\omega); \cdots; \tilde{t}_{r_2 j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_{r_2}(\omega)) \succeq_{r_2\langle 2\rangle} \mathbf{0}, \;\; j = 1, 2, \cdots, m \\
& (\tilde{t}_{(r_2+1)j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_{(r_2+1)}(\omega); \cdots; \tilde{t}_{(r_2+s_2)j}; \boldsymbol{x}_j - \tilde{\boldsymbol{a}}_{(r_2+s_2)}(\omega)) \succeq_{s_2\langle 1\rangle} \mathbf{0}, \;\; j = 1, 2, \cdots, m \\
& (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jl}; \boldsymbol{x}_j - \boldsymbol{x}_l) \succeq_{(l-j)\langle 2\rangle} \mathbf{0}, \;\; j = 1, 2, \cdots, l-1 \\
& (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jm}; \boldsymbol{x}_j - \boldsymbol{x}_m) \succeq_{(m-j)\langle 1\rangle} \mathbf{0}, \;\; j = l+1, 2, \cdots, m-1
\end{array}
$$

and

$$\mathbb{E}[Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega)] := \int_{\Omega} Q(\boldsymbol{x}_1; \cdots; \boldsymbol{x}_m, \omega) P(d\omega).$$

where $\hat{w}_{jj'}$ is the weight associated with the new facilities j' and j for $j' = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, m$.

5.3 Deterministic DERFLPs—A 0-1DMOCP model

We consider the discrete version of the problem by assuming that the new facilities $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m$ need to be placed at specific locations and not in any place in 2- or 3- (or higher) dimensional space. Let the points $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_k \in \mathbb{R}^n$ represent these specific locations where $k \geq m$. So, we add the constraint $\boldsymbol{x}_i \in \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_k\}$ for $i = 1, 2, \cdots, m$. Clearly, for $i = 1, 2, \cdots, m$, the above constraint can be replaced by the following linear and binary constraints:

$$
\begin{array}{l}
\boldsymbol{x}_i = \boldsymbol{v}_1\, y_{i1} + \boldsymbol{v}_2\, y_{i2} + \cdots + \boldsymbol{v}_k\, y_{ik}, \\
y_{i1} + y_{i2} + \cdots + y_{ik} = 1, \text{ and} \\
\boldsymbol{y}_i = (y_{i1}; y_{i2}; \cdots; y_{ik}) \in \{0, 1\}^k.
\end{array}
$$

We also assume that we cannot place more than one facility at each location. Consequently, we add the following constraints:

$$(1; y_{1l}; y_{2l}; \cdots; y_{ml}) \succeq_{\langle 1\rangle} \mathbf{0}, \;\; \text{for } \; l = 1, 2, \cdots, k.$$

If there is no interaction between the new facilities, then the DMOCP model (16) becomes the following 0-1DMOCP model:

$$
\begin{array}{ll}
\min & \sum_{j=1}^{m} \sum_{i=1}^{r+s} w_{ij}\ t_{ij} \\
\text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{rj}; \boldsymbol{x}_j - \boldsymbol{a}_r) \succeq_{r\langle 2\rangle} \mathbf{0}, \ \ j = 1, 2, \cdots, m \\
& (t_{(r+1)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+1}; \cdots; t_{(r+s)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+s}) \succeq_{s\langle 1\rangle} \mathbf{0}, \ \ j = 1, 2, \cdots, m \\
& \boldsymbol{x}_i = \boldsymbol{v}_1\ y_{i1} + \boldsymbol{v}_2\ y_{i2} + \cdots + \boldsymbol{v}_k\ y_{ik}, \ \ i = 1, 2, \cdots, m \\
& (1; y_{1l}; y_{2l}; \cdots; y_{ml}) \succeq_{\langle 1\rangle} \mathbf{0}, \ \ \text{for} \ \ l = 1, 2, \cdots, k \\
& \mathbf{1}^{\mathsf{T}}\ \boldsymbol{y}_i = 1, \ \boldsymbol{y}_i \in \{0, 1\}^k, \ \ i = 1, 2, \cdots, m
\end{array}
$$

If interaction exists among the new facilities, then the DMOCP model (17) becomes the following 0-1DMOCP model:

$$
\begin{array}{ll}
\min & \sum_{j=1}^{m} \sum_{i=1}^{r+s} w_{ij}\ t_{ij} + \sum_{j=2}^{m} \sum_{j'=1}^{j-1} \hat{w}_{jj'} \hat{t}_{jj'} \\
\text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{rj}; \boldsymbol{x}_j - \boldsymbol{a}_r) \succeq_{r\langle 2\rangle} \mathbf{0}, \ \ j = 1, 2, \cdots, m \\
& (t_{(r+1)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+1}; \cdots; t_{(r+s)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+s}) \succeq_{s\langle 1\rangle} \mathbf{0}, \ \ j = 1, 2, \cdots, m \\
& (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jl}; \boldsymbol{x}_j - \boldsymbol{x}_l) \succeq_{(l-j)\langle 2\rangle} \mathbf{0}, \ \ j = 1, 2, \cdots, l-1 \\
& (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jm}; \boldsymbol{x}_j - \boldsymbol{x}_m) \succeq_{(m-j)\langle 1\rangle} \mathbf{0}, \ \ j = l+1, 2, \cdots, m-1 \\
& \boldsymbol{x}_i = \boldsymbol{v}_1\ y_{i1} + \boldsymbol{v}_2\ y_{i2} + \cdots + \boldsymbol{v}_k\ y_{ik}, \ \ i = 1, 2, \cdots, m \\
& (1; y_{1l}; y_{2l}; \cdots; y_{ml}) \succeq_{\langle 1\rangle} \mathbf{0}, \ \ \text{for} \ \ l = 1, 2, \cdots, k \\
& \mathbf{1}^{\mathsf{T}}\ \boldsymbol{y}_i = 1, \ \boldsymbol{y}_i \in \{0, 1\}^k, \ \ i = 1, 2, \cdots, m
\end{array}
$$

For $l = 1, 2, \cdots, k$, let $z_l = 1$ if the location $\boldsymbol{v}_i$ is chosen, and 0 otherwise. Then, we can go further, and consider more assumptions: Let $k_1, k_2, k_3, k_4 \in [1, k]$ be integers such that $k_1 \leq k_2$ and $k_3 \leq k_4$. If we must choose at most k_1 of the locations $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{k_2}$, then we impose the constraints:

$$(k_1; z_1; z_2; \cdots; z_{k2}) \succeq_{\langle 1\rangle} \mathbf{0}, \text{ and } \boldsymbol{z} \in \{0, 1\}^k.$$

If we must choose at most k_1 of the locations $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{k_2}$, or at most k_3 of the locations $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{k_4}$, then we impose the constraints:

$$(k_1 f; z_1; z_2; \cdots; z_{k_2}) \succeq_{\langle 1\rangle} \mathbf{0}, \ (k_3(1-f); z_1; z_2; \cdots; z_{k_4}) \succeq_{\langle 1\rangle} \mathbf{0}, \ \boldsymbol{z} \in \{0, 1\}^k, \text{ and } f \in \{0, 1\}.$$

5.4 Deterministic ERFLPs with integrality constraints—A DMIMOCP model

In some problems we may need the locations to have integer-valued coordinates. In most cities, streets are laid out on a grid, so that city is subdivided into small numbered blocks that are square or rectangular. In this case, usually the decision maker needs the new facility to be placed at the corners of the city blocks. Thus, for each $i \in \Delta \subset \{1, 2, \cdots, m\}$, let us assume that the variable $\boldsymbol{x}_i$ lies in the hyper-rectangle $\Xi_i^n \equiv \{\boldsymbol{x}_i : \boldsymbol{\zeta}_i \leq \boldsymbol{x}_i \leq \boldsymbol{\eta}_i, \boldsymbol{\zeta}_i \in \mathbb{R}^n, \boldsymbol{\eta}_i \in \mathbb{R}^n\}$ and has to be integer-valued, i.e. $\boldsymbol{x}_i$ must be in the grid $\Xi_i^n \bigcap \mathbb{Z}^n$. Thus, if there is no interaction between the new facilities, then instead of solving the DMOCP model (16), we solve the following DMIMOCP model:

$$
\begin{array}{ll}
\min & \sum_{j=1}^{m} \sum_{i=1}^{r+s} w_{ij}\ t_{ij} + \sum_{j=2}^{m} \sum_{j'=1}^{j-1} \hat{w}_{jj'} \hat{t}_{jj'} \\[2ex]
\text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{rj}; \boldsymbol{x}_j - \boldsymbol{a}_r) \succeq_{r\langle 2\rangle} \boldsymbol{0},\ \ j = 1, 2, \cdots, m \\[2ex]
& (t_{(r+1)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+1}; \cdots; t_{(r+s)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+s}) \succeq_{s\langle 1\rangle} \boldsymbol{0},\ \ j = 1, 2, \cdots, m \\[2ex]
& \boldsymbol{x}_k \in \Xi_k^n \bigcap \mathbb{Z}^n,\ k \in \Delta
\end{array}
$$

If interaction exists among the new facilities, then instead of solving the DMOCP model (17), we solve the following DMIMOCP model:

$$
\begin{array}{ll}
\min & \sum_{j=1}^{m} \sum_{i=1}^{r+s} w_{ij}\ t_{ij} + \sum_{j=2}^{m} \sum_{j'=1}^{j-1} \hat{w}_{jj'} \hat{t}_{jj'} \\[2ex]
\text{s.t.} & (t_{1j}; \boldsymbol{x}_j - \boldsymbol{a}_1; \cdots; t_{rj}; \boldsymbol{x}_j - \boldsymbol{a}_r) \succeq_{r\langle 2\rangle} \boldsymbol{0},\ \ j = 1, 2, \cdots, m \\[2ex]
& (t_{(r+1)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+1}; \cdots; t_{(r+s)j}; \boldsymbol{x}_j - \boldsymbol{a}_{r+s}) \succeq_{s\langle 1\rangle} \boldsymbol{0},\ \ j = 1, 2, \cdots, m \\[2ex]
& (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jl}; \boldsymbol{x}_j - \boldsymbol{x}_l) \succeq_{(l-j)\langle 2\rangle} \boldsymbol{0},\ \ j = 1, 2, \cdots, l-1 \\[2ex]
& (\hat{t}_{j(j+1)}; \boldsymbol{x}_j - \boldsymbol{x}_{j+1}; \cdots; \hat{t}_{jm}; \boldsymbol{x}_j - \boldsymbol{x}_m) \succeq_{(m-j)\langle 1\rangle} \boldsymbol{0},\ \ j = l+1, 2, \cdots, m-1 \\[2ex]
& \boldsymbol{x}_k \in \Xi_k^n \bigcap \mathbb{Z}^n,\ k \in \Delta
\end{array}
$$

Finally, we mention that it is useful to consider the stochastic version of DERFLPs to obtain a 0-1SMOCP model, and to consider also, parallelly, the stochastic version of ERFLPs with integrality constraints to obtain an SMIMOCP model.

6 Concluding remarks

We conclude the paper by indicating possible directions for future research.

As indicated in Figure 1, the new classes of optimization problems DPOCPs, SPOCPs, DMIPOCPSs, DMOCPs, SMOCPs and DMIMOCPs introduced in this paper are natural extensions of deterministic, stochastic and mixed integer second-order cone programs. It is interesting to note that semidefinite programs and multi-order cone programs both include second-order cone programs

as a special case. In §5 we presented an application leading to multi-order cone programs. It is interesting to investigate other applicational settings leading to (deterministic, stochastic and mixed integer) multi-order cone programs. Development of algorithms for such multi-order cone programs which in turn will benefit from a duality theory is equally interesting and important. The authors are currently exploring these research directions.

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